TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 233, 1977

THE CLOSED LEAF INDEX OF FOLIATED MANIFOLDS

BY

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ABSTRACT. For M a closed, connected, oriented 3-manifold, a topological invariant is computed from the cohomology ring $H^*(M; \mathbf{Z})$ that provides an upper bound to the number of topologically distinct types of closed leaves any smooth transversely oriented foliation of M can contain. In general, this upper bound is best possible.

Introduction. Let M be a closed, connected, oriented n-manifold, F a smooth transversely oriented foliation of codimension one on M. Throughout this paper, the term "foliated n-manifold" will mean such a pair (M, F).

DEFINITION. The *closed leaf index* of F is the number $\gamma(F)$ of distinct topological types of closed leaves in F.

It is easy to see that $\gamma(F)$ is finite. By a theorem of S. P. Novikov [3], all foliations of S^3 have $\gamma(F) = 1$, and those of $S^1 \times S^2$ have $\gamma(F) \geq 1$. It is also known [2] that foliations of T^3 satisfy $0 \leq \gamma(F) \leq 2$ and that those of $S^1 \times S^2$ satisfy $1 \leq \gamma(F) \leq 2$, each of these possibilities being realized by a suitable foliation. By [1], the greatest lower bound of $\gamma(F)$ on any closed 3-manifold M is always 0 or 1. It is a result of P. Schweitzer [5] that, for $n \geq 5$, every n-manifold that can be foliated admits a C^0 foliation F (with C^∞ leaves) having $\gamma(F) = 0$. In this paper we will produce a fairly severe upper bound to the closed leaf index of foliated 3-manifolds. Analogous but weaker results will be obtained in higher dimensions.

DEFINITION. The symbol $\alpha(M)$ denotes the largest integer such that some basis $x_1, \ldots, x_r$ of $H^1(M; \mathbf{Z}) = \mathbf{Z}^r$ satisfies $x_i \cup x_j = 0$, $1 \leq i,j \leq \alpha(M)$.

For instance, $\alpha(S^3) = 0$, $\alpha(S^1 \times S^2) = 1$, and $\alpha(T^3) = 1$. Indeed, we will compute α for all oriented S^1-bundles over closed oriented surfaces (§3).

It is convenient to define a strictly increasing function $\varphi: \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$\varphi(k) = \begin{cases} k+1, & k = 0, 1, \\ 3k-2, & k \geq 2. \end{cases}$$

THEOREM A. *If (M, F) is a foliated 3-manifold, then $\gamma(F) \leq \varphi(\alpha(M))$.*

THEOREM B. *For each integer $k \geq 0$, there is a foliated 3-manifold (M_k, F_k) such that $\alpha(M_k) = k$ and $\gamma(F_k) = \varphi(k)$.*

Received by the editors May 14, 1975 and, in revised form, March 29, 1976.
AMS (MOS) subject classifications (1970). Primary 57D30; Secondary 57D95.

We do not know, however, whether Theorem A gives the best possible upper bound for $\gamma(F)$ on every 3-manifold. We conjecture that this fails, in fact, for the "least twisted" nontrivial oriented S^1-bundle over a closed, oriented surface of genus $g > 0$.

In higher dimensions it is not reasonable to expect such sharp information on $\gamma(F)$, but the following invariants are manageable.

DEFINITION. The *homology index* of F is the number $h(F)$ of classes in $H_{n-1}(M^n; \mathbf{Z})$, distinct modulo sign, that can be represented by closed leaves of F. The mod 2 homology index $h_2(F)$ is the number of distinct classes in $H_{n-1}(M^n; \mathbf{Z}_2)$ so represented.

DEFINITION. The *Euler index* of F is the number $e(F)$ of distinct integers $|\chi(L)|$ for closed leaves L of F.

On even dimensional manifolds, the Euler index is always 0 or 1; hence this invariant is of real significance only on odd dimensional manifolds. For foliated 3-manifolds, the equality $\gamma(F) = e(F)$ is immediate from the classification of closed surfaces and the Reeb stability theorem, so Theorem A and Theorem B are specializations respectively of Theorem A′ and Theorem B′ below.

THEOREM A′. *If (M, F) is a foliated n-manifold, then $e(F) \leq h(F) = h_2(F) \leq \varphi(\alpha(M))$.*

THEOREM B′. *For each pair of integers $(n, k) \neq (4, 0)$ with $n \geq 3$ and $k \geq 0$, there is a foliated n-manifold (M_k^n, F_k) with $\alpha(M_k^n) = k$ and $h(F_k) = \varphi(k)$. If n is odd, then also $\gamma(F_k) \geq e(F_k) = \varphi(k)$.*

The nonexistence of (M_0^4, F_0) is seen by noting that $H^1(M_0^4) = 0$. By Poincaré duality the relation $\chi(M_0^4) \geq 2$ must hold and, consequently, there can be no foliation of M_0^4 of codimension one.

Theorem A′ involves a series of topological and combinatorial considerations. Theorem B′ is a consequence of recent deep results of W. Thurston [6]. The authors are grateful to Thurston for preprints of his work.

1. Submanifold systems of codimension one. Let M be a closed, connected, oriented n-manifold. A set $\mathfrak{L} = \{L_1, \ldots, L_r\}$ of disjoint, imbedded, closed, connected, oriented submanifolds of M of codimension one will be called a *submanifold system*. In this section we assemble a number of facts about such systems, applications being made later to foliations.

We denote by $[L_i]$ the element of $H_{n-1}(M)$ represented by the fundamental cycle of L_i, and by $[\mathfrak{L}]$ the set $\{[L_1], \ldots, [L_r]\}$. Here and elsewhere, unless otherwise specified, all homology and cohomology will have integral coefficients. Note that $H_{n-1}(M) \cong H^1(M)$ and that this group is free abelian by the universal coefficient theorem.

For the usual reasons, any two maximal linearly independent subsets of $[\mathfrak{L}]$ have the same cardinality, namely the rank of $\text{span}_Z[\mathfrak{L}]$. The geometric interpretation of this fact is that any two subsets of $\mathfrak{L}$, maximal with respect to the property of not disconnecting M, have the same cardinality. We suppose that $\{[L_1], \ldots, [L_k]\}$ is a maximal linearly independent subset and define $\text{rank}(\mathfrak{L}) = k$.

LEMMA 1.1. *The set* $\{[L_1], \ldots, [L_k]\}$ *spans a direct summand of* $H_{n-1}(M)$. *An element* $x \in H_{n-1}(M)$ *which is linearly dependent on* $\{[L_1], \ldots, [L_k]\}$ *can be represented by a closed, connected, oriented submanifold* $L \subset M$ *disjoint from* L_i, $1 \leq i \leq k$, *if and only if* $x = \varepsilon_1[L_1] + \cdots + \varepsilon_k[L_k]$ *where each* $\varepsilon_i \in \{0, 1, -1\}$.

PROOF. Since $W = M - \bigcup_{i=1}^k L_i$ is an open connected subspace of M, we can let $\sigma_1, \ldots, \sigma_k$ be disjoint smoothly imbedded circles in M transverse to $\bigcup_{i=1}^k L_i$ with $\sigma_i \cap L_j$ a set of δ_{ij} points, $1 \leq i, j \leq k$. By suitably orienting these circles we obtain the homology intersection products $[\sigma_i] * [L_j] = \delta_{ij}$. In particular, $\{[L_1], \ldots, [L_k]\}$ spans a direct summand of $H_{n-1}(M)$.

If $L \subset W$ is a closed, connected, oriented submanifold of M which does not disconnect W, then the above argument shows that $[L]$ is linearly independent of $\{[L_1], \ldots, [L_k]\}$. But if L does disconnect W, then it is clear that $[L] = \sum_{i=1}^k \varepsilon_i [L_i]$, where each $\varepsilon_i \in \{0, 1, -1\}$.

If $x = \varepsilon_1[L_1] + \cdots + \varepsilon_k[L_k]$ and each $\varepsilon_i \in \{0, 1, -1\}$, we can suppose

$$x = [L_1] + \cdots + [L_h] - [L_{h+1}] - \cdots - [L_{h+q}]$$

(where either h or q may be 0). Let L_i' be obtained by displacing L_i a distance $\varepsilon > 0$ along the positive normal trajectories to L_i, $1 \leq i \leq h$, and along the negative normal trajectories if $h + 1 \leq i \leq h + q$. Using the fact that W is connected, we construct simple arcs τ_i: $[0, 1] \to M$ meeting the following specifications. No τ_i meets an L_j, $1 \leq j \leq k$, and the interior of no τ_i meets an L_j', $1 \leq j \leq h + q$. For $1 \leq i \leq h - 1$, the arc τ_i joins the positive side of L_i' to the positive side of L_{i+1}'. The arc τ_h joins the positive side of L_h' to the negative side of L_{h+1}'. For $h + 1 \leq i \leq h + q - 1$, the arc τ_i joins the negative side of L_i' to the negative side of L_{i+1}'. The connected sum of the manifolds L_i' via small tubes along these arcs is a closed, connected, oriented manifold L meeting no L_j, $1 \leq j \leq k$, such that $x = [L]$. □

In particular, we can now write $[L_j] = \sum_{i=1}^k \varepsilon_j^i [L_i]$, $1 \leq j \leq r$, where each $\varepsilon_j^i \in \{0, 1, -1\}$. Let E_j denote the coefficient k-tuple $(\varepsilon_j^1, \varepsilon_j^2, \ldots, \varepsilon_j^k)$ of $[L_j]$.

LEMMA 1.2. *If* $\{[L_p], [L_q]\}$ *is linearly independent, then* E_p *and* E_q *do not have exactly the same zero entries.*

PROOF. We can assume $p, q > k$. The set consisting of L_p together with those L_i for which $\varepsilon_p^i \neq 0$ (say $1 \leq i \leq h \leq k$ after suitable renumbering) disconnects the manifold into two components, W_1 and W_2, with common boundary $L_p \cup L_1 \cup \cdots \cup L_h$. The manifold L_q lies in one of these components, say W_1, and if E_q has exactly the same zero entries as E_p, then L_q separates W_1 into two components, one of which has boundary $L_q \cup L_1 \cup \cdots \cup L_h$. It follows that the other component has boundary $L_p \cup L_q$, contradicting the linear independence of $\{[L_p], [L_q]\}$. □

DEFINITION. $\mathfrak{L}$ is an *admissible system* if $[\mathfrak{L}]$ is pairwise linearly independent (or, equivalently, if $M - (L_i \cup L_j)$ is connected, $1 \leq i, j \leq r$).

REMARK. Denote by $[L_i]_2$ the element of $H_{n-1}(M; \mathbf{Z}_2)$ carried by L_i and by $[\mathfrak{L}]_2$ the set of these classes. The force of (1.2) is that, for $\mathfrak{L}$ admissible, $[\mathfrak{L}]$ and $[\mathfrak{L}]_2$ have the same cardinality.

Let $\mathfrak{L}$ be an admissible system. We can modify each L_i by attaching small handles $S^{n-2} \times I$ to L_i within Euclidean neighborhoods of M. If L_i' denotes the resulting submanifold, then $[L_i'] = [L_i]$ and the resulting admissible system $\mathfrak{L}'$ has exactly the same separation relations in M as did $\mathfrak{L}$. Furthermore, if $U_1, \ldots, U_q$ are the components of $M - \bigcup L_i$ and $U_1', \ldots, U_q'$ the corresponding components of $M - \bigcup L_i'$, then each U_j' is obtained from U_j by adjoining and/or cutting out solid handles. If a solid handle was adjoined, then $\chi(\overline{U_j'}) = \chi(\overline{U_j}) - 1$. If n is even and a solid handle was cut out, then $\chi(\overline{U_j'}) = \chi(\overline{U_j}) + 1$.

DEFINITION. $\mathfrak{L}'$ as above is said to be a *surgical modification* of $\mathfrak{L}$.

The principal results of this section will be the following propositions.

PROPOSITION 1.3. *If $\chi(M) = 0$ and $\mathfrak{L} = \{L_1, \ldots, L_r\}$ is an admissible system, then suitable surgical modifications on $\mathfrak{L}$ produce an admissible system $\mathfrak{L}' = \{L_1', \ldots, L_r'\}$ with the following properties*:

(1) *The tangent bundle to $\bigcup_{i=1}^r L_i'$ extends to an $(n-1)$-plane field on M,*

(2) *$H^1(L_i; \mathbf{R}) \neq 0$, $1 \leq i \leq r$,*

(3) *if n is odd, $\{\chi(L_1'), \ldots, \chi(L_r')\}$ is a set of distinct negative integers.*

PROPOSITION 1.4. *If $\mathfrak{L} = \{L_1, \ldots, L_r\}$ is an admissible system, then $r \leq \varphi(\alpha(M)) - 1$.*

The second of these propositions is the heart of the proof of Theorem A′, while the first, together with Thurston's results [6], makes possible a very straightforward construction of the foliated n-manifolds (M_k^n, F_k) for Theorem B′.

First we will need a technical lemma. The notation $E \cdot N$ in the statement of the lemma denotes the usual dot product of k-tuples.

LEMMA 1.5. *Let $\mathcal{E}^k = \{(\varepsilon_1, \ldots, \varepsilon_k)\colon \varepsilon_i \in \{0, 1, -1\}\}$. Given any $R > 0$, any integer $k \geq 1$, and any $\eta = (\eta_1, \ldots, \eta_k) \in (\mathbf{Z}_2)^k$, there exists $N = (n_1, \ldots, n_k) \in \mathbf{Z}^k$ with all $n_i < 0$ such that, for all $E, E' \in \mathcal{E}^k$,*

(1) $E \cdot N = E' \cdot N$ *if and only if* $E = E'$,
(2) $|E \cdot N| > R$ *if* $E \neq (0, \ldots, 0)$,
(3) $N \equiv \eta \bmod 2$.

PROOF. This is trivial for $k = 1$, so we argue by induction on k. Suppose $N = (n_1, \ldots, n_k)$ satisfies the conditions. Choose $E^* \in \mathcal{E}^k$ such that $|E^* \cdot N|$ is maximal. Choose $n_{k+1} < 0$ with the desired parity and such that $|n_{k+1}| > 2|E^* \cdot N|$. Let $N' = (n_1, \ldots, n_{k+1}) \in \mathbf{Z}^{k+1}$. If $\sum_{i=1}^{k+1} \varepsilon_i n_i = \sum_{i=1}^{k+1} \varepsilon_i' n_i$, ε_i, $\varepsilon_i' \in \{0, 1, -1\}$, then

$$(\varepsilon_{k+1}' - \varepsilon_{k+1}) n_{k+1} = \sum_{i=1}^{k} \varepsilon_i n_i - \sum_{i=1}^{k} \varepsilon_i' n_i = E \cdot N - E' \cdot N.$$

We consider the only possible cases:
(1) $|\varepsilon_{k+1}' - \varepsilon_{k+1}| = 1$,
(2) $|\varepsilon_{k+1}' - \varepsilon_{k+1}| = 2$,
(3) $\varepsilon_{k+1}' = \varepsilon_{k+1}$ and $E \cdot N = E' \cdot N$.
In case (1), $|n_{k+1}| = |E \cdot N - E' \cdot N| \leqslant 2|E^* \cdot N|$ and this contradicts the assumption that $|n_{k+1}| > 2|E^* \cdot N|$. In case (2), $|n_{k+1}| \leqslant |E^* \cdot N|$, again a contradiction. Thus case (3) alone is possible. By the inductive hypothesis, $E = E'$ and so $\varepsilon_i = \varepsilon_i'$, $1 \leqslant i \leqslant k + 1$. Furthermore, if $E'' = (\varepsilon_1'', \ldots, \varepsilon_{k+1}'') \in \mathcal{E}^{k+1}$, we claim $|E'' \cdot N'| > R$. Indeed, if $\varepsilon_{k+1}'' = 0$, this comes from the inductive hypothesis and, if $\varepsilon_{k+1}'' \neq 0$, then

$$|E'' \cdot N'| = \left| \sum_{i=1}^{k} \varepsilon_i'' n_i \pm n_{k+1} \right| \geqslant |n_{k+1}| - \left| \sum_{i=1}^{k} \varepsilon_i'' n_i \right|$$
$$> 2|E^* \cdot N| - |E^* \cdot N| = |E^* \cdot N| > R. \quad \square$$

We also need the following fairly well-known fact.

LEMMA 1.6. *If W is a compact connected n-manifold, v a vector field defined along ∂W and transverse to ∂W, ∂W_+ the part of ∂W along which v points out of W, ∂W_- the part along which v points inward, then v extends to a nowhere zero vector field on W if and only if $\chi(W) = \chi(\partial W_+) = \chi(\partial W_-)$. Equivalently, for n even, this holds precisely when $\chi(W) = 0$, and for n odd, it holds precisely when $\chi(\partial W_+) = \chi(\partial W_-)$.*

Indeed, v always extends to $\tilde{v}$ with only isolated nondegenerate singularities in $\operatorname{int}(W)$ and, W being connected, the singularities can be eliminated if the sum of their indices $\iota(\tilde{v}) = 0$. The condition that $\iota(\tilde{v}) = 0$ is exactly the condition in (1.6), as is seen by standard relative Hopf index formulas (a very general formula is given by C. Pugh [4]).

PROOF OF (1.3). To begin with, note that adding a single handle to L_i produces L_i' with $H^1(L_1'; \mathbf{R}) \neq 0$ and this property is never lost by adding more handles. This is easily established via the Mayer-Vietoris sequence. Thus, without loss of generality, we can suppose that $H^1(L_i; \mathbf{R}) \neq 0$, $1 \leq i \leq r$, and this property is never lost during the surgical modification, so property (2) is assured.

For the rest of the argument we must distinguish the cases in which n is odd or even. First suppose that n is even. Let $U_1, \ldots, U_q$ be the components of $M - \bigcup_{i=1}^r L_i$. Each $\overline{U}_j$ is a manifold with boundary $\partial \overline{U}_j$ equal to the union of some of the manifolds L_i. If $L_{i_1}, \ldots, L_{i_p}$ are in $\operatorname{int}(\overline{U}_j)$, then $\overline{U}_j - \bigcup_{m=1}^p L_{i_m}$ is connected and we produce a compact connected manifold W_j by cutting $\overline{U}_j$ open along each L_{i_m}. Note that $\chi(W_j) = \chi(\overline{U}_j)$. Also $0 = \chi(M) = \sum_{j=1}^q \chi(\overline{U}_j)$ since the boundary components of every $\overline{U}_j$ are odd dimensional, hence have vanishing Euler characteristic.

Find a sequence $U_{n_1}, \ldots, U_{n_p}$ such that each U_i appears at least once in the sequence and such that $\partial \overline{U}_{n_j}$ and $\partial \overline{U}_{n_{j+1}}$ have a common component L_{n_j}, $1 \leq j \leq p-1$. (Repetitions have to be allowed. For instance, $\partial \overline{U}_1$ might contain every L_i, which would force such a sequence as $U_1, U_2, U_1, U_3, U_1, \ldots$.) If $\chi(\overline{U}_{n_1}) \neq 0$, add a suitable number of handles to L_{n_1} (into $\overline{U}_{n_1}$ if $\chi(\overline{U}_{n_1}) > 0$, into $\overline{U}_{n_2}$ if $\chi(\overline{U}_{n_1}) < 0$) so that the new $\chi(\overline{U}_{n_1}) = 0$. Proceed in this way through the sequence. Since some U_{n_j} may equal some $U_{n_{j-i}}$, the good work already done on $U_{n_{j-i}}$ may be undone in passing from $U_{n_{j-1}}$ to U_{n_j}, but if $j < p$, this will be remedied in passing from U_{n_j} to $U_{n_{j+1}}$. When one finally arrives at U_{n_p}, every $U_{n_j} \neq U_{n_p}$ satisfies $\chi(\overline{U}_{n_j}) = 0$. Assume $U_{n_p} = U_q$ and observe that

$$0 = \chi(M) = \sum_{j=1}^{q} \chi(\overline{U}_j) = \chi(\overline{U}_q).$$

Thus, without loss of generality, we suppose $\mathfrak{L}$ is such that each $\chi(W_j) = 0$. Let v be the unit normal field along $\bigcup_{i=1}^r L_i$. This defines a field v_j along ∂W_j; hence (1.6) implies that v extends to a nonsingular field on all of M. Property (1) follows and, since n is even, property (3) is irrelevant.

Next suppose that n is odd. Surgical modification reduces the Euler characteristic of L_i by 2 for each handle added to L_i. In particular, this allows us to assume that $\chi(L_i) < 0$, $1 \leq i \leq r$. Remark also that for any compact odd-dimensional manifold W, $\chi(\partial W)$ is an even number since ∂W is cobordant to zero. Let $\{[L_1], \ldots, [L_k]\}$ be a maximal linearly independent subset of $[\mathfrak{L}]$. If $[L_i] = \sum_{j=1}^k \varepsilon_i^j [L_j]$ then L_i, together with those L_j for which $\varepsilon_i^j \neq 0$, cobound in M, so

$$\chi(L_i) \equiv \sum_{j=1}^{k} \varepsilon_i^j \chi(L_j) \mod 2.$$

Let $R > |\chi(L_i)|$, $1 \leq i \leq r$, and let $N = (n_1, \ldots, n_k)$ satisfy (1.5) with $n_j \equiv \chi(L_j) \bmod 2$, $1 \leq j \leq k$. Let $\sigma_1, \ldots, \sigma_k$ be circles in M such that $[\sigma_i] * [L_j] = \delta_{ij}$, $1 \leq i, j \leq k$. Let $\gamma = \sum_{j=1}^{k} n_j[\sigma_j]$. Then, if $E_i = (\varepsilon_i^1, \ldots, \varepsilon_i^k)$ denotes the coefficient k-tuple of $[L_i]$, we see that $\gamma * [L_i] = E_i \cdot N$. Since $E_i \neq \pm E_j$ for $i \neq j$, (1.5) implies that $|\gamma * [L_i]|$, $1 \leq i \leq k$, are distinct nonzero integers. Changing the orientation of L_i, if necessary, we can assume all $\gamma * [L_i]$ are negative. By the above,

$$|\gamma * [L_i]| > |\chi(L_i)|, \qquad \gamma * [L_i] \equiv \chi(L_i) \bmod 2, \qquad 1 \leq i \leq r.$$

Thus, by surgical modification, we can produce $\mathfrak{L}'$ such that $\gamma * [L_i] = \gamma * [L_i'] = \chi(L_i') < 0$, $1 \leq i \leq r$, and these integers are distinct.

Let $\xi \in H^{n-1}(M)$ be the Poincaré dual of γ. Then, $\xi([L_i]) = \int_{L_i'} \xi = \chi(L_i')$. Let v be the unit normal field along $\bigcup_{i=1}^{r} L_i'$ directed in the positive sense (relative to the orientation of L_i'). Let $U_1, \ldots, U_q$ be the components of $M - \bigcup_{i=1}^{r} L_i'$ and let ξ_i be the restriction of ξ to $\overline{U}_i$. On $\partial \overline{U}_i$, understood as oriented inward, ξ_i integrates to 0; hence on $\partial \overline{U}_{i+}$, oriented outward, ξ_i integrates to the same value as on $\partial \overline{U}_{i-}$, oriented inward. That is,

$$\chi(\partial U_{i+}) = \chi(\partial U_{i-}), \qquad 1 \leq i \leq q.$$

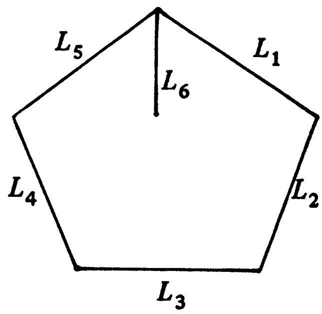
Let W_i be the compact connected manifold obtained by cutting $\overline{U}_i$ open along each $L_j' \subset \operatorname{int}(\overline{U}_i)$. Then v defines v_i along ∂W_i and transverse to ∂W_i. Each $L_j' \subset \operatorname{int}(\overline{U}_i)$ determines two diffeomorphic boundary components of W_i, one in ∂W_{i+} and one in ∂W_{i-}. Thus $\chi(\partial W_{i+}) = \chi(\partial W_{i-})$, $1 \leq i \leq q$, so v_i extends to a nonsingular vector field over W_i (by (1.6)); hence v extends to a nonsingular vector field over M. This defines the desired extension of the tangent bundle of $\bigcup_{i=1}^{r} L_i'$, hence establishes property (1). Properties (2) and (3) have already been assured, so the proof of (1.3) is complete. □

The proof of (1.4) will proceed by a sequence of lemmas. We will be interested only in the separation properties of $\mathfrak{L}$ in M, so all homological arguments will be carried out with $\mathbf{Z}_2$-coefficients. This is legitimate by reason of (1.2) and the remark following that lemma. Thus, rank($\mathfrak{L}$) is dim(span$[\mathfrak{L}]_2$) and the coefficient k-tuple E_j for $[L_j]_2$ will be an element of $(\mathbf{Z}_2)^k$.

DEFINITION. If $\mathfrak{L}$ and $\mathfrak{L}'$ are admissible systems, then $\mathfrak{L} \leq \mathfrak{L}'$ means $\mathfrak{L} \subset \mathfrak{L}'$ and rank($\mathfrak{L}$) = rank($\mathfrak{L}'$).

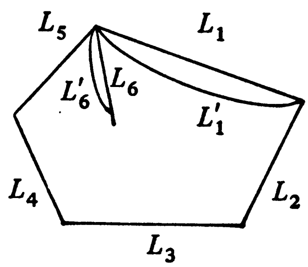
DEFINITION. An admissible system $\mathfrak{L} = \{L_1, \ldots, L_r\}$ is *polygonal* if each L_i occurs as a boundary component of some $\overline{U}_j$ (hence of exactly two) where $U_1, \ldots, U_q$ is the set of components of $M - \bigcup_{i=1}^{r} L_i$.

The term "polygonal" is due to a heuristic procedure by which each $\overline{U}_j$ is represented by a diagram such as

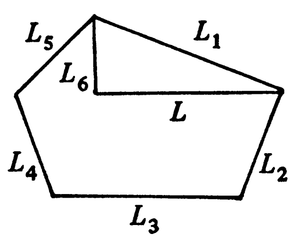


where $\partial \bar{U}_j = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$ and $L_6 \subset \text{int}(\bar{U}_j)$. The word "polygonal" indicates that each $\bar{U}_j$ is represented by a simple closed polygon, hence has no L_i in the interior.

In the above example, produce L_1' and $L_6' \subset \text{int}(U_j')$ by displacing L_1 and L_6 a small distance $\varepsilon > 0$ along normal trajectories. A schematic representation of this situation would be



since L_i and L_i' cobound for $i = 1, 6$. A connected sum of L_1' and L_6' along a small tube lying entirely in $\text{int}(\bar{U}_j)$ and missing L_6 produces $L \subset \text{int}(\bar{U}_j)$ such that $L \cap L_6 = \varnothing$ and $L \cup L_6$ disconnects $\bar{U}_j$. The schematic representation of the final state of affairs would be



Finally, if $[L]_2 \neq [L_i]_2$ for every $L_i \in \mathfrak{L}$, this construction produces $\mathfrak{L}' = \{L_1, \ldots, L_r, L\}$ with $\mathfrak{L} < \mathfrak{L}'$. This idea is exploited in the following lemma.

LEMMA 1.7. *If $\mathfrak{L}$ is an admissible system of rank at least 2, then $\mathfrak{L} < \mathfrak{L}'$ for some admissible polygonal system $\mathfrak{L}'$.*

PROOF. As usual, arrange that $\{[L_1]_2, \ldots, [L_k]_2\}$ is a maximal linearly independent subset of $[\mathfrak{L}]_2$. If $L_i \subset \text{int}(\bar{U}_j)$, then $L_1 \cup \cdots \cup L_k \cup L_i$ cannot separate M, hence $i \leqslant k$. Arrange that $\{L_1, \ldots, L_p\}, p \leqslant k$, is the subset of those L_i each of which is contained in the interior of some $\overline{U}_j$. Thus every $E_j = (\varepsilon_j^1, \ldots, \varepsilon_j^k)$ has $\varepsilon_j^i = 0$ for $1 \leqslant i \leqslant p$. The system $\mathfrak{L}$ is polygonal if and only if $p = 0$.

If $p = k$, then $k = r \geqslant 2$ and by (1.1) the class $x = [L_1]_2 + \cdots + [L_r]_2$ can be represented by connected orientable $L_{r+1} \subset M$ disjoint from every L_i, $1 \leqslant i \leqslant r$, $[L_{r+1}]_2$ being linearly independent of $[L_i]_2$, $1 \leqslant i \leqslant r$. $\mathfrak{L}' = \{L_1, \ldots, L_r, L_{r+1}\}$ is admissible of rank $k = r$ and is also polygonal.

If $0 < p < k$, let U_i be the component such that $L_p \subset \operatorname{int}(\overline{U}_i)$. Let $L_s \subset \partial \overline{U}_i$. As in the remarks preceding the lemma, displace both L_s and L_p slightly and form a connected sum so as to produce $L_{r+1} \subset \operatorname{int}(\overline{U}_i)$ with

$$L_{r+1} \cap L_j = \varnothing, \qquad 1 \leqslant j \leqslant r,$$

$$[L_{r+1}]_2 = [L_p]_2 + [L_s]_2.$$

This gives an admissible system $\{L_1, \ldots, L_{r+1}\}$ of rank k in which $\{L_1, \ldots, L_{p-1}\}$ is the subset of those L_i contained in some $\operatorname{int}(\overline{U}_j)$. This process, repeated finitely often, produces the desired polygonal system $\mathfrak{L}'$. □

Remark that polygonal systems must have rank at least 2, hence the necessity of the restriction on rank($\mathfrak{L}$) in (1.7). This in turn is the reason why $\varphi(k)$ must be defined by a different formula when $k = 0, 1$.

Definition. An admissible system is *triangular* if it is polygonal and if each $\partial \overline{U}_j$ has exactly three components.

Lemma 1.8. *Let $\mathfrak{L}$ be an admissible polygonal system. Then $\mathfrak{L} \leqslant \mathfrak{L}'$ for some admissible triangular system.*

Proof. If some $\partial(\overline{U}_j)$ has more than three components (fewer are impossible by the definition of an admissible system), let L_1, L_2, L_3, L_4 be distinct components of $\partial \overline{U}_j$. If both $\{[L_1]_2, [L_2]_2, [L_3]_2\}$ and $\{[L_2]_2, [L_3]_2, [L_4]_2\}$ are linearly dependent sets, then $[L_1]_2 = [L_2]_2 + [L_3]_2$ and $[L_4]_2 = [L_2]_2 + [L_3]_2$, and this would contradict $[L_1]_2 \neq [L_4]_2$. Thus we can assume that $\{[L_1]_2, [L_2]_2, [L_3]_2\}$ is a linearly independent set. We claim that either $[L_1]_2 + [L_2]_2$ or $[L_1]_2 + [L_3]_2$ is not in $[\mathfrak{L}]_2$. Indeed, if one can find L_p, $L_q \in \mathfrak{L}$ such that $[L_p]_2 = [L_1]_2 + [L_2]_2$, $[L_q]_2 = [L_1]_2 + [L_3]_2$, then $p, q \neq 1, 2, 3$ and $p \neq q$. Thus $\{L_1, L_2, L_3, L_p, L_q\}$ is an admissible system of rank 3. The complement in M of $L_1 \cup L_2 \cup L_p$ has two components. Neither L_3 nor L_q individually disconnects the component in which it lies, but $L_1 \cup L_3 \cup L_q$ disconnects M, so L_3 and L_q must lie in the same component of $M - L_1 - L_2 - L_p$ and together disconnect it into two components, one of which must have boundary $L_1 \cup L_3 \cup L_q$ (because all other possibilities lead to a linear dependence between $[L_1]_2$ and $[L_2]_2$). Thus $M - (L_1 \cup L_2 \cup L_3 \cup L_p \cup L_q)$

has components W_1, W_2, W_3 with

$$\partial \overline{W}_1 = L_1 \cup L_3 \cup L_q,$$

$$\partial \overline{W}_2 = L_3 \cup L_q \cup L_2 \cup L_p,$$

$$\partial \overline{W}_3 = L_1 \cup L_2 \cup L_p.$$

Since $\mathfrak{L}$ contains this admissible system, $\overline{U}_j$ must be contained in one of $\overline{W}_1$, $\overline{W}_2$, $\overline{W}_3$. But $\overline{U}_j \subset \overline{W}_1$ contradicts $L_2 \not\subset \overline{W}_1$, $\overline{U}_j \subset \overline{W}_2$ contradicts $L_1 \not\subset \overline{W}_2$, and $\overline{U}_j \subset \overline{W}_3$ contradicts $L_3 \not\subset \overline{W}_3$. Thus we can assume that no $L_p \in \mathfrak{L}$ has $[L_p]_2 = [L_1]_2 + [L_2]_2$. Thus, displace both L_1 and L_2 to the interior of $\overline{U}_j$ and form their connected sum, producing $L_{r+1} \subset U_j$ which separates $\overline{U}_j$ into two components, one with boundary $L_{r+1} \cup L_1 \cup L_2$, the other with one fewer boundary component than $\overline{U}_j$ had. By the above, $\{L_1, \ldots, L_r, L_{r+1}\}$ is admissible, polygonal, and has the same rank as $\mathfrak{L}$. Finite iteration of this process will finally yield the desired triangular system. □

Thus every admissible $\mathfrak{L}$ can be extended to an admissible triangular system $\mathfrak{L}'$ of the same rank. The next project is to produce a formula for the number of elements in $\mathfrak{L}'$ in terms of the integer rank($\mathfrak{L}'$).

LEMMA 1.9. *Let W be a compact, orientable, connected manifold and suppose that ∂W has h components, $h \geq 3$. Let $L_1, \ldots, L_p \subset \mathrm{int}(W)$ be disjoint, closed, connected, orientable submanifolds of* int(W) *of codimension one. Suppose that $\{L_1, \ldots, L_s\}$ is a maximal subset such that $W - \bigcup_{i=1}^{s} L_i$ is connected. Suppose that, for each component U of $W - \bigcup_{i=1}^{p} L_i$, the manifold $\partial \overline{U}$ has three components and that every L_i occurs as some boundary component for some U. Then $p = h + 3s - 3$.*

PROOF. We do a double induction on $h \geq 3$ and $s \geq 0$. If $h = 3$ and $s = 0$, we claim that $p = 0$. Otherwise, L_1 separates W into two components one of which, say W', must be bounded by L_1 and a single component of ∂W. If there is any $L_i \subset W'$, then, since $s = 0$, we again get a component of $W - (L_1 \cup L_i)$ with two boundary components. Proceeding in this way, we finally see that some component of $W - \bigcup_{i=1}^{p} L_i$ has just two boundary components, contrary to our hypothesis.

Suppose the assertion is true for $s = 0$ and for all h such that $3 \leq h < m$. If ∂W has m components and $s = 0$, then L_1 divides W into two components each of which has at least three boundary components (for the same reason as above). If the number of boundary components in each is h_1 and h_2 respectively, then $3 \leq h_i < m$, $i = 1, 2$, and $h_1 + h_2 = m + 2$. But each of $L_2, \ldots, L_p$ belongs to one or another component of $W - L_1$ and these components are divided into triangular components. The inductive hypothesis implies that

$$p - 1 = h_1 - 3 + h_2 - 3 = h_1 + h_2 - 6 = m - 4$$

and so $p = m - 3$ as desired.

Finally, assume the assertion for $0 \leqslant s < n$ and for all h. Suppose $\{L_1, \ldots, L_n\}$ is a maximal subset not separating W. Necessarily $n < p$ since, otherwise, $h = 3$, $p = 0$, and so $n = 0$. Thus, the set $\{L_1, \ldots, L_n, L_{n+1}\}$ separates W into two components W_1 and W_2. We can suppose that the common boundary is $\partial W_1 \cap \partial W_2 = L_1 \cup \cdots \cup L_q \cup L_{n+1}$ where $0 \leqslant q \leqslant n$. We can even suppose $q > 0$ since, if this were impossible to arrange, none of $L_1, \ldots, L_n$ could ever occur as a boundary component of any component of $W - \bigcup_{i=1}^{p} L_i$, contradicting the assumption that $n > 0$. Suppose that s_1 of $\{L_{q+1}, \ldots, L_n\}$ fall into W_1 and that s_2 of them fall into W_2. Evidently these s_i manifolds are maximal nonseparating in W_i and $s_i < n$, $i = 1, 2$. Let h_i be the number of boundary components of W_i and p_i the number of $L_j \subset \operatorname{int}(W_i)$, $i = 1, 2$. By the inductive hypothesis $p_i = h_i + 3s_i - 3$, $i = 1, 2$. But $h_1 + h_2 = h + 2q + 2$, $s_1 + s_2 + q = n$, $p_1 + p_2 = p - q - 1$, and so

$$\begin{aligned} p = p_1 + p_2 + q + 1 &= (h_1 + 3s_1 - 3) + (h_2 + 3s_2 - 3) + q + 1 \\ &= (h + 2q + 2) + 3s_1 + 3s_2 + q - 5 \\ &= h + 3(s_1 + s_2 + q) - 3 = h + 3n - 3. \quad \square \end{aligned}$$

LEMMA 1.10. *Let* $\mathfrak{L} = \{L_1, \ldots, L_r\}$ *be an admissible triangular system of rank* k. *Then* $r = 3k - 3$.

PROOF. Let U be any component of $M - \bigcup_{i=1}^{r} L_i$ and let $\partial \bar{U} = L_1 \cup L_2 \cup L_r$. Let $W_1 = \bar{U}$ and $W_2 = M - U$. These are compact manifolds each with three boundary components. Also, W_2 is connected since $\{[L_1]_2, [L_2]_2, [L_r]_2\}$ is pairwise linearly independent. The set $\{[L_1]_2, [L_2]_2\}$ extends to a maximal independent set $\{[L_1]_2, \ldots, [L_k]_2\}$. If $i \neq 1, 2, r$, then $L_i \subset \operatorname{int}(W_2)$ and $\{L_3, \ldots, L_k\}$ is a maximal nonseparating subset in W_2. Thus, by (1.9), $r - 3 = 3 + 3(k - 2) - 3 = 3k - 6$ and so $r = 3k - 3$. $\square$

We now prove (1.4). If $\mathfrak{L} = \{L_1, \ldots, L_r\}$ is an admissible system of rank k, let $\{[L_1], \ldots, [L_k]\}$ be a maximal independent subset. We emphasize the return to integral homology. By (1.1), this set spans a direct summand of $H_{n-1}(M)$. If x_i denotes the Poincaré dual of $[L_i]$, $1 \leqslant i \leqslant k$, then $\{x_1, \ldots, x_k\}$ extends to a basis of $H^1(M)$. But $x_i \cup x_j$ is the Poincaré dual of $[L_i] * [L_j] = 0$; hence $k \leqslant \alpha(M)$. If $k = 0$, then $r = 0$ and there is nothing to prove. If $k = 1$, then $r = 1$ and $1 \leqslant \alpha(M)$, so $r = \varphi(1) - 1 \leqslant \varphi(\alpha(M)) - 1$. If $k \geqslant 2$, then (1.7), (1.8) and (1.10) together imply

$$r \leqslant 3k - 3 = \varphi(k) - 1 \leqslant \varphi(\alpha(M)) - 1. \quad \square$$

2. Applications to foliated n-manifolds. It is easy to prove Theorem A′ as a corollary to (1.4). Let (M, F) be a foliated n-manifold with $h(F) = r$. Let

$\{L_1, \ldots, L_r\}$ be a set of closed leaves of F such that $\{[L_1], \ldots, [L_r]\}$ are distinct modulo sign. If no $[L_i] = 0$, then $\{L_1, \ldots, L_r\}$ is an admissible system. If, say, $[L_r] = 0$, then $\{L_1, \ldots, L_{r-1}\}$ is admissible. In the first case, $r \leq \varphi(\alpha(M)) - 1$, and in the second, $r - 1 \leq \varphi(\alpha(M)) - 1$. In any case, $h(F) \leq \varphi(\alpha(M))$. Also, by (1.2), $h_2(F) = h(F)$.

If L and L' are closed leaves of F with $|\chi(L)| \neq |\chi(L')|$, then $[L] \neq \pm[L']$. Indeed, let $c \in H^{n-1}(M)$ be the Euler class of the tangent bundle to F. Then, if $[L] = \pm[L']$, we would have

$$\pm \chi(L) = c[L] = \pm c[L'] = \pm\chi(L'),$$

a contradiction. It follows that $e(F) \leq h(F)$ and the proof of Theorem A′ is complete. As remarked in the introduction, Theorem A is a trivial consequence when $\dim(M) = 3$. □

Definition. An admissible system $\mathfrak{L}$ is *integrable* if there is a foliation F of M with each $L_i \in \mathfrak{L}$ as a leaf. (Note that the orientation of L_i as a leaf of F may not be the same as that of $L_i \in \mathfrak{L}$.)

Theorem 2.1. *If $\chi(M) = 0$ and $\mathfrak{L}$ is admissible, then a suitable surgical modification on $\mathfrak{L}$ produces an integrable $\mathfrak{L}'$. If, in addition,* $\dim(M)$ *is odd, it can be arranged that the Euler characteristics of the elements of $\mathfrak{L}'$ form a set of distinct negative integers.*

Proof. We allow $\mathfrak{L} = \varnothing$, in which case $\operatorname{rank}(\mathfrak{L}) = 0$. Produce $\mathfrak{L}'$ satisfying (1.3). Then, by Thurston ([6], relative version if $\operatorname{rank}(\mathfrak{L}) \neq 0$), there is a foliation of M having each $L'_i \in \mathfrak{L}'$ as a leaf. □

Corollary 2.2. *If $\chi(M) = 0$ and M supports an admissible system of rank k, then M admits a foliation F with $h(F) \geq \varphi(k)$. (Here we allow an empty admissible system; hence $k = 0$). If* $\dim(M)$ *is odd, such F can also be chosen so that $e(F) \geq \varphi(k)$.*

Proof. If $k \geq 2$, then by (1.7), (1.8), and (1.10) there is an admissible system $\mathfrak{L}$ of cardinality $3k - 3$. In any case, there is such a system $\mathfrak{L}$ of cardinality $\varphi(k) - 1$. By (2.1) we can assume the existence of a foliation F having each $L \in \mathfrak{L}$ as a leaf. Also, on odd-dimensional manifolds it can be arranged that the Euler characteristics $\chi(L)$, $L \in \mathfrak{L}$, form a set of distinct negative integers. By the Reeb stability theorem we can assume that some leaf is noncompact; hence standard methods show that there is a closed transversal to F missing every leaf $L \in \mathfrak{L}$. Modifying F along this transversal [7] produces a foliation F' in which each element of $\mathfrak{L}$ is a leaf, but which also has a homologically trivial closed leaf with Euler characteristic zero. Thus $h(F') \geq \varphi(k)$, and on odd dimensional manifolds M, one also has $e(F') \geq \varphi(k)$. □

In order to prove Theorem B′, we will now construct a suitable M^n_k and apply the above corollary together with Theorem A′.

First suppose that n is odd and at least equal to 3. Let $M_0^n = S^n$ and $M_1^n = T^n$. Evidently $\alpha(S^n) = 0$ and $\alpha(T^n) = 1$. For $k \geqslant 2$, let W_k^n be the complement in D^n of k disjoint open balls with closures in the interior of D^n and let M_k^n be the double of W_k^n. By the above corollary and the obvious fact that M_k^n supports an admissible system $\{L_1, \ldots, L_k\}$ of rank k, we see that M_k^n supports a foliation F_k with $h(F_k) \geqslant e(F_k) \geqslant \varphi(k)$.

If n is even and at least equal to 4, and if $k > 0$, set $M_k^n = M_k^{n-1} \times S^1$. If $\{L_1, \ldots, L_k\}$ is an admissible system of rank k on M_k^{n-1}, then the manifolds $L_i' = L_i \times S^1 \subset M_k^n$, $1 \leqslant i \leqslant k$, form an admissible system of rank k. For the case $k = 0$, set $M_0^6 = S^3 \times S^3$ and $M_0^n = S^3 \times S^3 \times S^{n-6}$, $n > 6$. In all cases, $\chi(M_k^n) = 0$ and the above corollary guarantees the existence of a foliation F_k with $h(F_k) \geqslant \varphi(k)$.

The following lemma together with Theorem A′ clearly completes the proof of Theorem B′. As remarked in the introduction, when $n = 3$ we get Theorem B as a special case.

LEMMA 2.3. *$\alpha(M_k^n) = k$ for all $n \geqslant 3$ and all $k \geqslant 0$ (excepting, of course, the nonexistent M_0^4).*

PROOF. For $k = 0, 1$ this is obvious. For n odd and $k \geqslant 2$, $\pi_i(W_k^n) = 0$, $0 \leqslant i \leqslant n - 2$, so the Hurewicz theorem gives

$$H_{n-1}(W_k^n) = \pi_{n-1}(W_k^n) = \mathbf{Z}^k.$$

There is an exact Mayer-Vietoris sequence

$$H_n(W_k^n) \oplus H_n(W_k^n) \to H_n(M_k^n) \to H_{n-1}(\partial W_k^n) \to H_{n-1}(W_k^n) \oplus H_{n-1}(W_k^n) \to H_{n-1}(M_k^n) \to H_{n-2}(\partial W_k^n).$$

But $H_n(W_k^n) = 0$, $H_n(M_k^n) = \mathbf{Z}$, $H_{n-1}(\partial W_k^n) = \mathbf{Z}^{k+1}$, $H_{n-1}(W_k^n) = \mathbf{Z}^k$ and $H_{n-2}(\partial W_k^n) = 0$, so we get

$$0 \to \mathbf{Z} \to \mathbf{Z}^{k+1} \to \mathbf{Z}^{2k} \to H_{n-1}(M_k^n) \to 0;$$

hence $0 \to \mathbf{Z}^k \to \mathbf{Z}^{2k} \to H_{n-1}(M_k^n) \to 0$. As earlier remarked, $H_{n-1}(M_k^n)$ is free abelian, so the sequence splits and $\mathbf{Z}^{2k} \cong \mathbf{Z}^k \oplus H_{n-1}(M_k^n)$. Therefore $H_{n-1}(M_k^n) \cong \mathbf{Z}^k$ for n odd. The admissible system $\{L_1, \ldots, L_k\}$ provides a free basis of $H_{n-1}(M_k^n)$ with all intersection products zero. By Poincaré duality, we get a basis $x_1, \ldots, x_k$ of $H^1(M_k^n)$ with all $x_i \cup x_j = 0$, so $\alpha(M_k^n) = k$.

If n is even, then

$$H_{n-1}(M_k^n) \cong H_{n-1}(M_k^{n-1}) \oplus H_{n-1}(M_k^{n-1}) \cong \mathbf{Z}^{k+1}.$$

Again we get classes $x_1, \ldots, x_k \in H^1(M_k^n)$ with all $x_i \cup x_j = 0$. These extend to a basis by adjunction of the class x_{k+1} defined by the projection map of M_k^n onto the factor S^1, and $x_i \cup x_{k+1} \neq 0$, $1 \leq i \leq k$. It is again clear that $\alpha(M_k^n) = k$. □

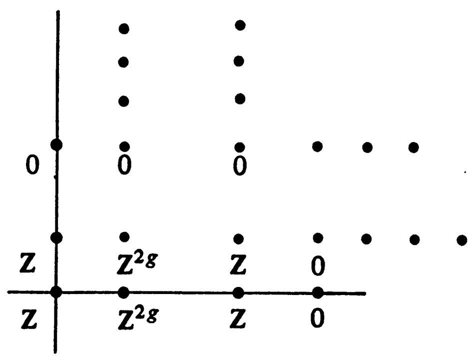
3. **A class of examples.** Let T_g denote the closed oriented surface of genus g. Let $E_{g,n}$ denote the total space of the oriented S^1-bundle over T_g having Euler class $n \in H^2(T_g) = \mathbf{Z}$. Since $E_{g,n} \cong E_{g,-n}$, we will always take $n \geq 0$. Remark, in particular, that $E_{0,0} = S^1 \times S^2$, $E_{0,1} = S^3$ and $E_{1,0} = T^3$.

We will compute the following.

$$\alpha(E_{g,n}) = \begin{cases} 1, & n = g = 0, \\ 2g, & n = 1, \\ g, & \text{otherwise.} \end{cases} \tag{3.1}$$

Indeed, $H^1(E_{0,0}) = \mathbf{Z}$, so the first equality is immediate.

Since the bundle $S^1 \hookrightarrow E_{g,n} \to_\pi T_g$ is orientable, the cohomology spectral sequence has ordinary coefficients. Since $H^1(T_g) = \mathbf{Z}^{2g}$, the E_2 stage takes the following form.



If $1 \in \mathbf{Z} = E_2^{0,1} = H^1(S^1)$ corresponds to the orientation of the fiber, then

$$d_2(1) = n \in \mathbf{Z} = E_2^{2,0} = H^2(T_g)$$

is the Euler class of the bundle. It is also clear that $E_3 = E_\infty$.

LEMMA 3.2. *$H^1(E_{g,1}) = \mathbf{Z}^{2g}$ and the cup products of elements in this group are all zero. Thus $\alpha(E_{g,1}) = 2g$.*

PROOF. In the spectral sequence, $d_2(1) = 1$ and so d_2: $E_2^{0,1} \to E_2^{2,0}$ is an isomorphism. Thus, $E_3^{0,1} = 0$ and $E_3^{1,0} = \mathbf{Z}^{2g}$. Since $E_3 = E_\infty$, we have $H^1(E_{g,1}) = \mathbf{Z}^{2g}$. Indeed, by the edge homomorphism, π^*: $H^1(T_g) \to H^1(E_{g,1})$ is an isomorphism. Since $E_3^{2,0} = 0$, the edge homomorphism also shows that π^*: $H^2(T_g) \to H^2(E_{g,1})$ is trivial. It follows that all cup products of elements in $H^1(E_{g,1})$ are trivial. □

LEMMA 3.3. *If $n > 1$, then $H^1(E_{g,n}) = \mathbf{Z}^{2g}$, $H^2(E_{g,n}) = \mathbf{Z}^{2g} \oplus \mathbf{Z}_n$, and there is a basis $x_1, \ldots, x_{2g}$ of $H^1(E_{g,n})$ such that*

(1) $x_i \cup x_{g+i} = x_j \cup x_{g+j}$ *is a generator of* $\mathbf{Z}_n \subset H^2(E_{g,n})$, $1 \leqslant i,j \leqslant g$,
(2) $x_i \cup x_j = 0$ *if* $|i-j| \neq g$.

PROOF. In the spectral sequence, $d_2(1) = n$, so $E_3^{0,1} = 0$, $E_3^{2,0} = \mathbf{Z}_n$, $E_3^{1,0} = E_3^{1,1} = \mathbf{Z}^{2g}$. By the edge homomorphism, $\pi^*: H^1(T_g) \to H^1(E_{g,n})$ is an isomorphism, the group being $\mathbf{Z}^{2g}$, and $\pi^*: H^2(T_g) \to H^2(E_{g,n})$ has image $\mathbf{Z}_n$. Also, there is an exact sequence

$$0 \to \mathbf{Z}_n \to H^2(E_{g,n}) \to \mathbf{Z}^{2g} \to 0$$

so $H^2(E_{g,n}) = \mathbf{Z}_n \oplus \mathbf{Z}^{2g}$. Finally, $H^1(T_g)$ has a basis $y_1, \dots, y_{2g}$ such that, for a generator $z \in H^2(T_g)$,

$$y_i \cup y_{g+i} = z, \qquad 1 \leqslant i \leqslant g$$

$$y_i \cup y_j = 0, \qquad |i-j| \neq g.$$

Setting $x_i = \pi^*(y_i)$ gives the desired basis of $H^1(E_{g,n})$. □

COROLLARY 3.4. *If* $n > 1$, $\alpha(E_{g,n}) = g$.

PROOF. Let $p > 1$ be a prime which divides n. The bilinear map $\mathbf{Z}^{2g} \times \mathbf{Z}^{2g} \to \mathbf{Z}_n$ given by

$$H^1(E_{g,n}) \times H^1(E_{g,n}) \xrightarrow{\cup} \mathbf{Z}_n \subset H^2(E_{g,n}),$$

when tensored with $\mathbf{Z}_p$, gives a bilinear map $\mathbf{Z}_p^{2g} \times \mathbf{Z}_p^{2g} \to \mathbf{Z}_p$. This is a bilinear form on the $\mathbf{Z}_p$-vector space $\mathbf{Z}_p^{2g}$ whose matrix, relative to the basis $x_1, \dots, x_{2g}$, is

$$J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

If $z_1, \dots, z_{2g}$ is another basis of $H^1(E_{g,n})$ and if $z_i \cup z_j = 0$, $1 \leqslant i, j \leqslant g+1$, then the matrix of the form becomes

$$P^*JP = \begin{bmatrix} 0_{g+1} & * \\ * & * \end{bmatrix}$$

where P is the nonsingular matrix corresponding to the change of basis. But J is nonsingular while the above matrix is clearly singular, so we have reached a contradiction. □

The manifolds $E_{g,0} = S^1 \times T_g$ must be handled a bit differently. We need the following lemma which is easily checked via the Künneth theorem.

LEMMA 3.5. $H^1(E_{g,0}) = \mathbf{Z}^{2g+1} = H^2(E_{g,0})$ *and there is a basis* $x_1, \dots, x_{2g}, y$ *of* $H^1(E_{g,0})$ *such that*
(1) $x_i \cup x_{g+i} = x_j \cup x_{g+j} = z$, $1 \leqslant i,j \leqslant g$,
(2) $x_1 \cup y, x_2 \cup y, \dots, x_{2g} \cup y, z$ *form a basis of* $H^2(E_{g,0})$,
(3) $x_i \cup x_j = 0$, $|i-j| \neq g$.

In particular, of course, $\alpha(E_{g,0}) \geq g$. We must prove the reverse inequality for all $g \geq 1$.

Let $V \subset H^1(E_{g,0}, \mathbf{Q})$ be the vector subspace spanned by $\{x_1, \ldots, x_{2g}\}$. Define the bilinear form $\varphi: V \times V \to \mathbf{Q}$ by

$$\varphi(v, w) \cdot z = v \cup w.$$

Then, (3.5) implies that the matrix of φ relative to the basis $\{x_1, \ldots, x_{2g}\}$ is

$$\Phi = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

This matrix is nonsingular. As in the proof of (3.4), the following will be enough to prove $\alpha(E_{g,0}) \leq g$.

LEMMA 3.6. *If $\alpha(E_{g,0}) > g$, there is a basis $\beta_1, \ldots, \beta_{2g}$ of V such that $\varphi(\beta_i, \beta_j) = 0$, $1 \leq i,j \leq g + 1$.*

PROOF. There will be linearly independent elements $z_1, \ldots, z_{g+1}$ of $H^1(E_{g,0}; \mathbf{Q})$ with $z_i \cup z_j = 0$, $1 \leq i, j \leq g + 1$. Express these elements in terms of the basis given in (3.5) as follows:

$$z_i = \sum_{j=1}^{2g} a_j^i x_j + a_{2g+1}^i y.$$

Then

$$0 = z_i \cup z_k = \left(\sum_{j=1}^{g} a_j^i a_{g+j}^k - a_{g+j}^i a_j^k \right) z + \sum_{j=1}^{2g} \left(a_j^i a_{2g+1}^k - a_j^k a_{2g+1}^i \right) x_j \cup y$$

and from (3.5) we can conclude $0 = \Sigma_{j=1}^{g}(a_j^i a_{g+j}^k - a_{g+j}^i a_j^k)$ and so, if we set

$$\beta_i = z_i - a_{2g+1}^i y = \sum_{j=1}^{2g} a_j^i x_j, \qquad 1 \leq i \leq g + 1,$$

we obtain $\beta_i \cup \beta_k = 0$, $1 \leq i,k \leq g + 1$. In order to show that $\{\beta_1, \ldots, \beta_{g+1}\}$ is linearly independent, it will be enough to show that y is linearly independent of $\{z_1, \ldots, z_{g+1}\}$. This will complete the proof.

Suppose $y = \Sigma_{i=1}^{g+1} b_i z_i$. Then

$$0 = \left(\sum b_i z_i \right) \cup z_k = y \cup z_k = \sum_{j=1}^{2g} a_j^k y \cup x_j, \qquad 1 \leq k \leq g + 1.$$

By (3.5) we conclude that all $a_j^k = 0$ for $j \leq 2g$, so $z_k = a_{2g+1}^k y$, $1 \leq k \leq g + 1$. Since we are assuming $g \geq 1$, this contradicts the linear independence of $\{z_1, \ldots, z_{g+1}\}$. □

Formula (3.1) is now completely proven. It is natural to ask whether Theorem A gives the best upper bound for $\gamma(F)$ on $E_{g,n}$. For $n \neq 1$, it is easy to show that this is true.

THEOREM 3.7. *If $n \neq 1$ and $(n, g) \neq (0, 0)$, then there is a foliation F of $E_{g,n}$ with $\gamma(F) = \varphi(g)$.*

PROOF. The surface T_g supports a set of disjointly imbedded circles $\sigma_1, \ldots, \sigma_g$ which together do not disconnect T_g. Thus $\{\pi^{-1}(\sigma_1), \ldots, \pi^{-1}(\sigma_g)\}$ is an admissible system in $E_{g,n}$ of rank g. By (2.2), there is a foliation F with $\gamma(F) \geq \varphi(g)$. If $n \neq 1$ and $(n, g) \neq (0, 0)$, (3.1) says that $\alpha(E_{g,n}) = g$; hence Theorem A says that $\gamma(F) \leq \varphi(g)$. □

We do not know whether $E_{g,1}$ admits a foliation F with $\gamma(F) = \varphi(2g)$, but we doubt it.

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